Few-nucleon systems in the Bethe-Salpeter approach

S. Bondarenko

BLTP, Joint Institute for Nuclear Research, Dubna, Russia

Experiments, motivation

Few-nucleon systems with high energy transfer:

– JLab Hall A Collaboration elastic lectron-deuteron sscattering at Q^2=0.7-6.0 (GeV/c)^2
Alexa L. C. et al. (Jefferson Lab Hall A), Phys. Rev. Lett. 82, 1999, 1374;

– deuteron electrodisintegration at Q^2=4.25 (GeV/c)^2 в JLab Hall C E12-10-003
Boeglin W. U. et al., arXiv:nucl-ex/1410.6770,
JEFFERSON-LAB-EXPERIMENT-E12-10-003;

– deep-inelastic scattering of the electron on three-nucleon systems at $E = 10.6$ GeV (Jefferson Lab MARATHON Coll.)
E12-10-103 *Abrams D. et al. (Jefferson Lab Hall A Tritium)*, Phys. Rev. Lett. 128, 2022, 132003;

...

Two-nucleon system. Bethe-Salpeter equation

$$T(p,p';P) = V(p,p';P) + \frac{i}{(2\pi)^4}\int d^4k\, V(p,k;P)\, G(k;P)\, T(k,p';P)$$

p', p - the relative four-momenta
P - the total four-momentum

$T(p,p';P)$ – two-nucleon t matrix
$V(p,p';P)$ – kernel of nucleon-nucleon interaction
$G(p;P)$ – free scalar two-particle propagator

$$G^{-1}(p;P) = \left[(P/2+p)^2 - m_N^2 + i\epsilon\right]\left[(P/2-p)^2 - m_N^2 + i\epsilon\right]$$

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable rank-one *Ansatz* for the kernel

$$V_L(p_0', |\mathbf{p}'|; p_0, |\boldsymbol{p}|; s) = \lambda^{[L]}(s) g^{[L]}(p_0', |\mathbf{p}'|) g^{[L]}(p_0, |\boldsymbol{p}|)$$

Solution for the T matrix

$$T_L(p_0', |\mathbf{p}'|; p_0, |\boldsymbol{p}|; s) = \tau(s)\, g^{[L]}(p_0', |\mathbf{p}'|)\, g^{[L]}(p_0, |\boldsymbol{p}|)$$

with

$$\Big[\tau(s)\Big]^{-1} = \Big[\lambda^{[L]}(s)\Big]^{-1} + h(s),$$

$$h(s) = \sum_{coupled\, L} h_L(s) = -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2\, d|\mathbf{k}| \sum_L [g^{[L]}(k_0, |\mathbf{k}|)]^2 S(k_0, |\mathbf{k}|; s)$$

$g^{[L]}$ - the model function, $\lambda^{[L'L]}(s)$ - a model parameter.

Relativistic Graz-II, rank-III kernel ($J = 1 :^3 S_1 -^3 D_1$ partial states)

$$g_1^{(S)}(p_0, |\boldsymbol{p}|) = \frac{1 - \gamma_1(p_0^2 - \boldsymbol{p}^2)}{(p_0^2 - \boldsymbol{p}^2 - \beta_{11}^2)^2},$$

$$g_2^{(S)}(p_0, \boldsymbol{p}) = -\frac{(p_0^2 - \boldsymbol{p}^2)}{(p_0^2 - \boldsymbol{p}^2 - \beta_{12}^2)^2},$$

$$g_3^{(D)}(p_0, |\boldsymbol{p}|) = \frac{(p_0^2 - \boldsymbol{p}^2)(1 - \gamma_2(p_0^2 - \boldsymbol{p}^2))}{(p_0^2 - \boldsymbol{p}^2 - \beta_{21}^2)(p_0^2 - \boldsymbol{p}^2 - \beta_{22}^2)^2},$$

$$g_1^{(D)}(p_0, |\boldsymbol{p}|) = g_2^{(D)}(p_0, |\boldsymbol{p}|) = g_3^{(S)}(p_0, |\boldsymbol{p}|) \equiv 0.$$

Table: Deuteron and low-energy properties scattering

	$p_D(\%)$	ϵ_D (MeV)	Q_D (Fm^{-2})	μ_D ($e/2m_N$)	$\rho_{D/S}$	r_0 (Fm)	a (Fm)
Graz II (рел.)	4	2.225	0.2484	0.8279	0.02408	1.7861	5.4188
Graz II (нерел.)	4.82	2.225	0.2812	0.8522	0.0274	1.78	5.42
Эксперимент		2.2246	0.286	0.8574	0.0263	1.759	5.424

Elastic $eD \to eD$ scattering

Structure functions and form factors:

$$A(q^2) = F_{\rm C}^2(q^2) + \frac{8}{9}\eta^2 F_{\rm Q}^2(q^2) + \frac{2}{3}\eta F_{\rm M}^2(q^2),$$
$$B(q^2) = \frac{4}{3}\eta(1+\eta) F_{\rm M}^2(q^2),$$

с $\eta = -q^2/4M^2$. Electric $F_{\rm C}(q^2)$, quadrupole $F_{\rm Q}(q^2)$ and magnetic $F_{\rm M}(q^2)$ form factors.

$$F_{\rm C}(0) = 1, \quad F_{\rm Q}(0) = M^2\, Q_{\rm D}, \quad F_{\rm M}(0) = \mu_{\rm D}\frac{M}{m},$$

где m_N - nuclon mass, $Q_{\rm D}$ quadrupole and $\mu_{\rm D}$ magnetic deuteron moments.

Deuteron currect matrix element

$$\langle D'\mathcal{M}'|j_\mu|D\mathcal{M}\rangle = -\,\xi^*_{\alpha\ \mathcal{M}'}(P')\,\xi_{\beta\ \mathcal{M}}(P)\,\Bigg[(P'+P)_\mu\Big(g^{\alpha\beta}F_1(q^2) - \frac{q^\alpha q^\beta}{2M^2}F_2(q^2)\Big) - (q^\alpha g^\beta_\mu - q^\beta g^\alpha_\mu)G_1(q^2)\Bigg],$$

with $\xi_{\mathcal{M}}(P)$ and $\xi^*_{\mathcal{M}'}(P')$ – polarization 4-vectors. Form factors $F_{1,2}(q^2)$, $G_1(q^2)$ and $F_{\mathrm{C}}(q^2)$ are related to $F_{\mathrm{Q}}(q^2)$, $F_{\mathrm{M}}(q^2)$.

Relativistic impulse approximation

$$\langle D'\mathcal{M}'|j_\mu|D\mathcal{M}\rangle = i\int\frac{\mathrm{d}p}{(2\pi)^4}\,\mathrm{Tr}\Big\{\bar{\chi}_{\mathcal{M}'}(P',p')\Gamma^{(S)}_\mu(q)\chi_{\mathcal{M}}(P,p)(P\cdot\gamma/2 - p\cdot\gamma + m)\Big\},$$

Lorentz transformations

$$P'_{lab} = \mathcal{L}P'_{rest} = \mathcal{L}(M, \mathbf{0}), \quad p'_{lab} = \mathcal{L}p'_{rest}$$

With $\mathcal{L}$:

$$\mathcal{L} = \begin{pmatrix} 1+2\eta & 0 & 0 & 2\sqrt{\eta}\sqrt{1+\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2\sqrt{\eta}\sqrt{1+\eta} & 0 & 0 & 1+2\eta \end{pmatrix}$$

$$\begin{aligned} p'_0 &= (1+2\eta)p_0 - 2\sqrt{\eta}\sqrt{1+\eta}p_z - M\eta, \\ p'_x &= p_x, \quad p'_y = p_y, \\ p'_z &= (1+2\eta)p_z - 2\sqrt{\eta}\sqrt{1+\eta}p_0 + M\sqrt{\eta}\sqrt{1+\eta}, \end{aligned}$$

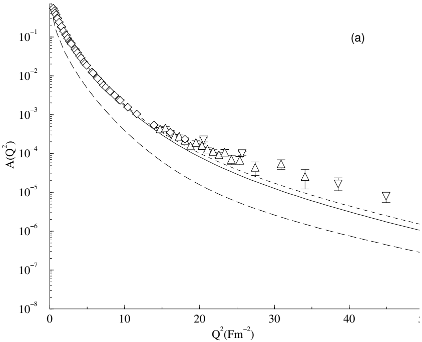
where p_0, p_x, p_y, p_z are the components of the p 4-vector in Lab system

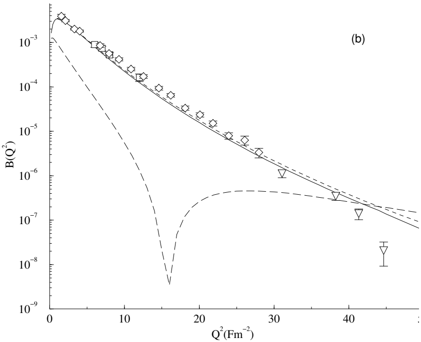
Nucleon form factors

– *dipole fit (DFF)*
– *modfied vector meson dominance model (MVMD)*
– *relativistic harmonic occilator model* (RHOM)
– *modified dipole fit* (MDFF1)
– *nine resonance model* (UAM)

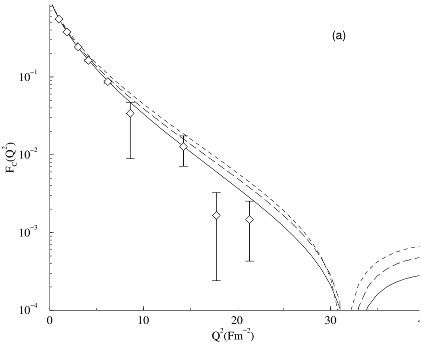
The long dashed line - results with taking into account “moving singularities”, the solid line - full relativistic calculations, the short dashed line - nonrelativistic ones.

(a) $A(q^2)$ structure function.
(b) $B(q^2)$ structure function.

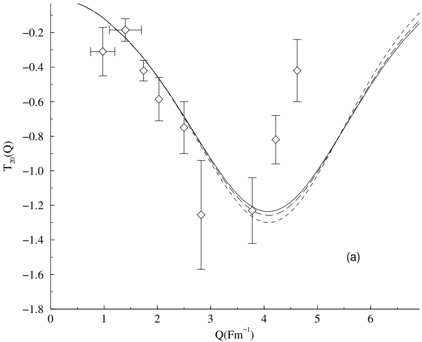




(a) Charge deuteron form factor $F_{\mathrm{C}}(q^2)$.
(b) Quadrupole form factor $F_{\mathrm{Q}}(q^2)$



(a) Tensor of polarization component $T_{20}(q^2)$.
(b) Tensor of polarization component $T_{21}(q^2)$.



Three-nucleon systems. Integral Faddeev equations

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - i \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

$T^{(i)}$ – full 3-body t-matrix components: $T = \sum_i T^{(i)}$
G_i – Green functions of the free particles (j and n) (индексы ijn представляют собой циклическую перестановку тройки (1,2,3)).
In scalar case their are:

$$G_i(k_j, k_n) = S(k_j)S(k_n) = (k_j^2 - m_N^2 + i\epsilon)^{-1}(k_n^2 - m_N^2 + i\epsilon)^{-1}$$

Separable kernel

For separable kernel:

$$t^{al}(p_0,p,k_0,k;s_q) = \sum_{ij} g_i^{al}(p_0,p)\tau_{ij}^{al}(s_q)g_j^{al}(k_0,k)$$

Three-nucleon amplitude:

$$\psi_{l\lambda L}^{a}(p_0,p,q_0,q;s) = \sum_{ij} g_{il}^{a}(p_0,p)\tau_{ijl}^{a}((\frac{2}{3}K - q_0^{'})^2 - q^{'2})\Phi_{jl\lambda L}^{a}(q_4,q),$$

where Φ can be found from the equation:

$$\Phi_{jl\lambda L}^{a}(q_0,q) = \frac{i}{4\pi^3}\sum_{b}\sum_{kn}\sum_{l^{'}\lambda^{'}}\int_{-\infty}^{\infty} dq_0^{'}\int_0^{\infty} q^{'2}dq^{'}\times$$

$$Z_{jkl\lambda l^{'}\lambda^{'}L}^{ab}(q_0,q;q_0^{'},q^{'};s)\frac{\tau_{knl^{'}}^{b}[(\frac{2}{3}\sqrt{s}-q_0^{'})^2-q^{'2}]}{(\frac{1}{3}\sqrt{s}-q_0^{'})^2-q^{'2}-m^2}\Phi_{nl^{'}\lambda^{'}L}^{b}(q_0^{'},q^{'}),$$

whith

$$Z_{jkl\lambda l^{'}\lambda^{'}L}^{ab}(q_0,q;q_0^{'},q^{'};s) = \Delta_l^a\Delta_{l^{'}}^b C^{ab}\int_{-1}^{1} dx K_{l\lambda l^{'}\lambda^{'}L}(q,q^{'},x)\times$$

Results for $3N$ system binding energy

One-rank kernel

p_D	${}^1S_0 - {}^3S_1$	${}^3D_1$	${}^3P_0$	${}^1P_1$	${}^3P_1$
4	9.221	9.294	9.314	9.287	9.271
5	8.819	8.909	8.928	8.903	8.889
6	8.442	8.545	8.562	8.540	8.527

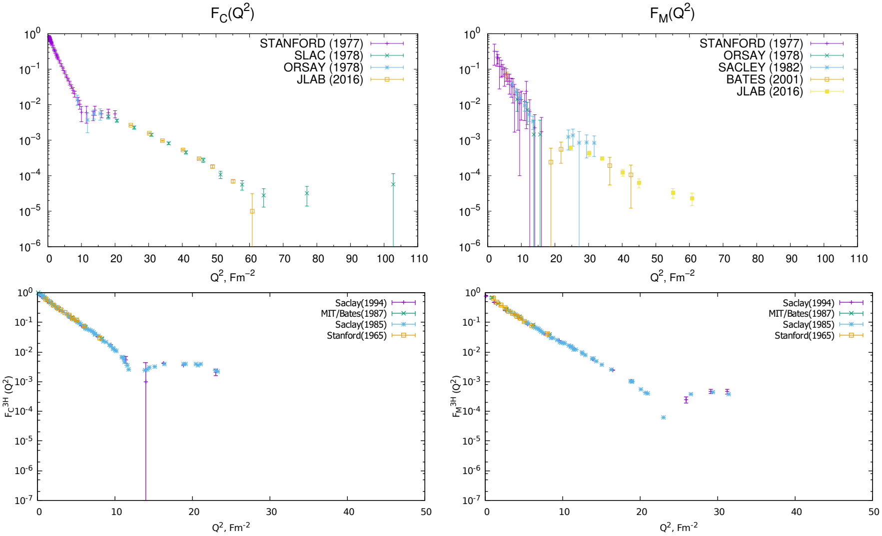
Graz-II kernel

Kernel	Nonrelativistic		Relativistic	
	${}^1S_0, {}^3S_1$	${}^1S_0, {}^3S_1, {}^3D_1$	${}^1S_0, {}^3S_1$	${}^1S_0, {}^3S_1, {}^3D_1$
Graz-II $p_D = 4$ %	8.372	8.334	8.628	8.617
Graz-II $p_D = 5$ %	7.964	7.934	8.223	8.217
Graz-II $p_D = 6$ %	7.569	7.548	7.832	7.831

Paris-1(2) kernel

Kernel	Nonrelativistic	Relativistic
Paris-1	7.245	7.545
Paris-2	7.183	7.379

Experimental data for $^3\mathrm{He}$, $^3\mathrm{H}$



Relativistic impulse approximation:

$$2F_C(^3He) = (2F_C^p + F_C^n)F_1 - \frac{2}{3}(F_C^p - F_C^n)F_2 + 2(F_n - F_p)F_3,$$

$$F_C(^3H) = (F_C^p + 2F_C^n)F_1 - \frac{2}{3}(F_C^n - F_C^p)F_2 + 2(F_p - F_n)F_3,$$

$$\mu(^3He)F_M(^3He) = \mu_n F_M^n F_1 - 2(\mu_p F_M^p + \mu_n F_M^n)F_2 + \frac{4}{3}(F_{\mathrm{M}}^p - F_{\mathrm{M}}^n)F_4,$$

$$\mu(^3H)F_M(^3H) = \mu_p F_M^p F_1 - 2(\mu_p F_M^p + \mu_n F_M^n)F_2 + \frac{4}{3}(F_{\mathrm{M}}^n - F_{\mathrm{M}}^p)F_4$$

Fumctions $F_1 - F_4$

$$F_i(Q) = \int d^4p \int d^4q\, G_1(k_1)G_2(k_2)G_3(k_3)\{f_i(p,q,Q)\}G_1'(k_1')$$

$$\begin{aligned}
f_1(p,q,Q) &= (\Psi_s(p,q)\Psi_s^*(p,q') + \Psi'(p,q)\Psi'^*(p,q') + \\
&\quad \Psi''(p,q)\Psi''^*(p,q') + \Psi_a(p,q)\Psi_a^*(p,q')),\\
f_2(p,q,Q) &= -3\Psi_s(p,q)\Psi_s'^*(p,q'),\\
f_3(p,q,Q) &= \Psi_a(p,q)\Psi''^*(p,q'),\\
f_4(p,q,Q) &= \Psi''(p,q)\Psi''^*(p,q')
\end{aligned}$$

where

$$G_i = (k_i^2 - m^2 + i\epsilon)^{-1}$$

$$k_1 = \frac{1}{3}K - q, \quad k_2 = \frac{1}{3}K + p + \frac{1}{2}q, \quad k_3 = \frac{1}{3}K - p + \frac{1}{2}q,$$

$$k_1' = \frac{1}{3}K' - q' = \frac{1}{3}K + Q - q, K' = K + Q \quad q' = q - \frac{2}{3}Q,$$

with K и K' are the total momenta for initial and final systems.

Breit system and Lorentz transformation

Breit system:

$$Q = (0, \mathbf{Q}), \qquad K = (E_B, -\frac{\mathbf{Q}}{2}), \qquad K' = (E_B, \frac{\mathbf{Q}}{2}),$$

$E_B = \sqrt{\mathbf{Q}^2/4 + s}$, $\sqrt{s} = M_t = 3m_N - E_t$.

Lorentz transformation

$$K = LK_{cm}, \qquad p = Lp_{cm}, \qquad q = Lq_{cm}.$$

In matrix form: $(\eta = \mathbf{Q}^2/4s)$:

$$L = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix} \tag{1}$$

$$K' = L^{-1}K'_{cm}, \qquad p' = L^{-1}p'_{cm}, \qquad q' = L^{-1}q'_{cm}.$$

$$q_0' = (1+2\eta)\, q_0 - 2\sqrt{\eta}\sqrt{1+\eta}\, q_z + \frac{2}{3}\sqrt{\eta}\, Q,$$
$$q_x' = q_x\,, \qquad q_y' = q_y,$$
$$q_z' = (1+2\eta)\, q_z - 2\sqrt{\eta}\sqrt{1+\eta}\, q_0 - \frac{2}{3}\sqrt{1+\eta}\, Q,$$

with $q_z = q\cos\theta_{qQ}$.

Static appriximation (SA)

Omit all terms propotional to $\sqrt{\eta}$:

$$q_0' = q_0, \qquad \mathbf{q}' = \mathbf{q} - \frac{2}{3}\mathbf{Q},$$

$$G_1'(q_0', q') \to \left[(\frac{1}{3}\sqrt{s} - q_0)^2 - \mathbf{q}^2 - \frac{2}{3}\mathbf{q}\cdot\mathbf{Q} - \frac{4}{9}\mathbf{Q}^2 - m_N^2 + i\epsilon \right]^{-1},$$
$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|),$$

$\mathbf{q}\cdot\mathbf{Q} = qQy$, $y = \cos\theta_{qQ}$.
In this case the Wick rotatio procedure is used $q_0 \to iq_4$.

Relativistic corrections

– full expressions for q' in $G_1'(k_1')$ and $\Psi(p,q')$ (boost corrections - BC):

$$G_1' = \Big[q_0^2 + \frac{2}{3}\sqrt{s}(1+4\eta)q_0 + 4\sqrt{1+\eta}\sqrt{s}\sqrt{\eta}q_z$$
$$-\frac{8}{3}\eta s x + \frac{1}{9}s - \mathbf{q}^2 - m_N^2 + i\epsilon\Big]^{-1},$$
$$\Psi_i(p_0,p,q_0',q') \to \Psi_i(p_0,p,q_0,|\mathbf{q} - \frac{2}{3}\mathbf{Q}|). \qquad (2)$$

– accounting the simple pole in III quadrant at $Q^2 < Q^2_{min} = -2\sqrt{s}/3$ $(3m_N - \sqrt{s})$
Then

$$\int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dq_0\, f(p_0,q_0) = -\int_{-\infty}^{\infty} dp_4 \int_{-\infty}^{\infty} dq_4\, f(ip_4,iq_4)$$
$$+2\pi \operatorname*{Res}_{q_0=q_0^{(2)}} \int_{-\infty}^{\infty} dp_4\, f(ip_4,q_0)$$

This is a pole contribution (PC).

– expansion contribution (EC)):

$$\Phi(iq_4', q') = \Phi(iq_4, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|)$$
$$+\left[C_{q_4}\frac{\partial}{\partial q_4}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|} + \left[C_q\frac{\partial}{\partial q}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|},$$

где

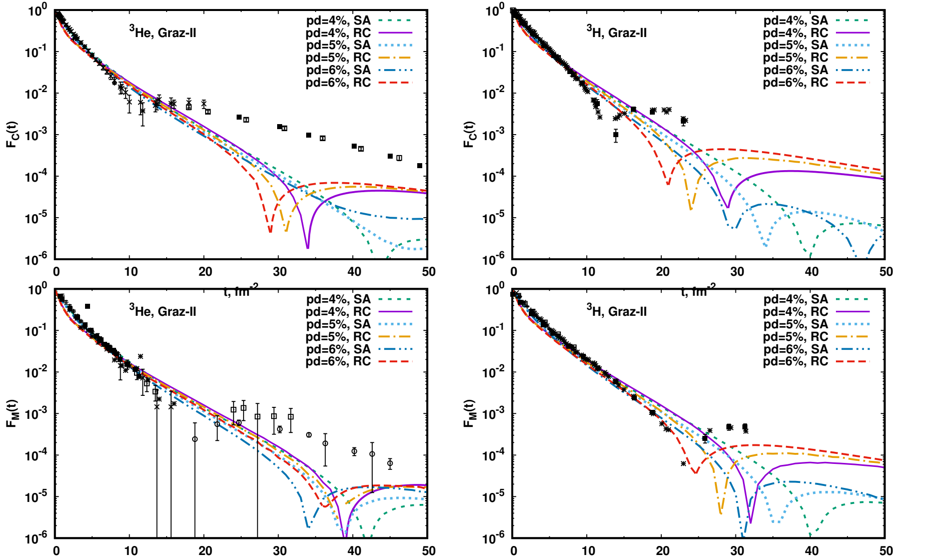
$$C_{q_4} = -i\left(2i\eta q_4 - 2\sqrt{1+\eta}\sqrt{\eta}qy + \frac{2}{3}\sqrt{\eta}Q\right),$$
$$C_q = \left(2\eta qy - 2i\sqrt{1+\eta}\sqrt{\eta}q_4 - \frac{2}{3}(\sqrt{\eta}-1)Q\right)y,$$

где $y = \cos(\mathbf{q}, \mathbf{Q})$.
Where

$$\Phi' = \int K'\Phi$$

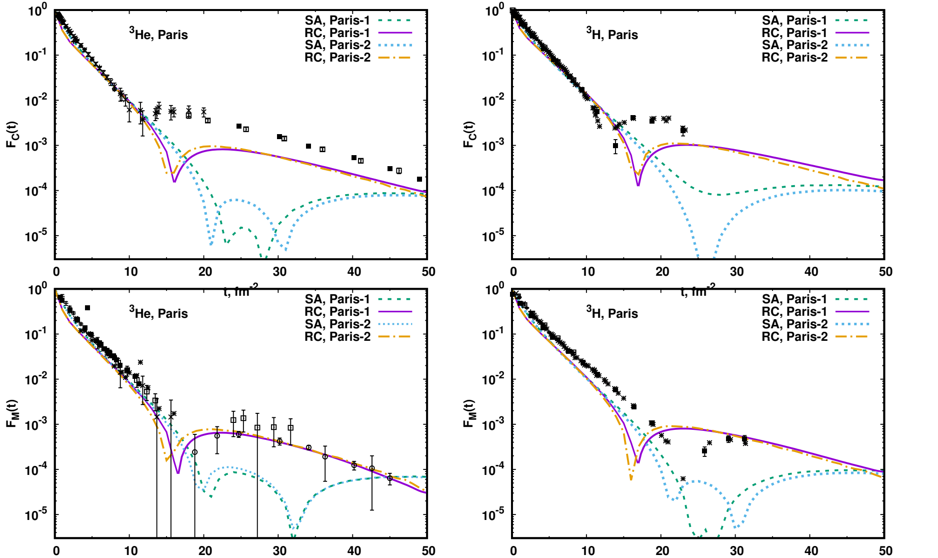
Results for multirank kernel Graz-II

(a) $F_{\rm C}(^3{\rm He})$ (b) $F_{\rm C}(^3{\rm H})$ (c) $F_{\rm M}(^3{\rm He})$ (d) $F_{\rm M}(^3{\rm H})$



Results for multirank kernel Paris

(a) $F_{\mathrm{C}}(^3\mathrm{He})$ (b) $F_{\mathrm{C}}(^3\mathrm{H})$ (c) $F_{\mathrm{M}}(^3\mathrm{He})$ (d) $F_{\mathrm{M}}(^3\mathrm{H})$



Conclusion

- the relativistic BS and BSF equations solved with separable multiramk kernels of NN interactions
- the charge, magnetic, quadroupole for the deuteron and the charge and magnetic for the 3N systems EM form factors were calculated
- the static approximation and relativistic corrections were investigated
- the relativistic corrections were found to be significant in describing the experimental data